ISSN 2229-5518

Bondage Numbers of Interval Graphs

K. Dhanalakshmi , B. Maheswari

Abstract-Interval graphs have drawn the attention of many researchers for over 30 years. They are extensively studied and revealed their practical relevance for modelling problems arising in the real world. In this paper we study various bondage numbers of an interval graph such as cobondage number, efficient bondage number, nonbondage number and find some bounds for these parameters.
Index Terms-Bondage number, Clique, Co-bondage Number Complement of a graph, dominating set, Efficient bondage Number, Interval Graph, Non bondage Number.
Subject Classification 68R10

1. Introduction

The theory of domination in graphs introduced by Ore [1] and Berge [2] is an emerging area of research in graph theory today. A survey on results and applications of dominating sets was presented by E.J.Cockyane and S.T. Hedentiemi [3]. A subset D of V is said to be a dominating set of G if every vertex in V \ D is adjacent to a vertex in D. The domination number γ of G is the minimum cardinality of a dominating set. The cobondage number of G is the minimum cardinality among all sets of edges E_1 in G^c for which $\gamma(G+E_1) < \gamma(G)$.This concept was introduced by V.R.Kulli *et al.* [4]

A set S of vertices in G is called an efficient dominating set if every vertex u in V \ S is adjacent to exactly one vertex in S. The efficient domination number is the minimum cardinality of an efficient dominating set. This concept was introduced by Cockayne *et al.* [5]. The concept of an efficient bondage number was introduced by V.R.Kulli *et al.* [6]. Let E_1 be the set of edges such that $\gamma_e(G-E_1) > \gamma_e(G)$. Then the efficient bondage number b_e of G is the minimum number of edges in E_1.

The nonbondage number $b_n(G)$ of a graph G is the maximum cardinality among all sets of edges $X \subseteq E$ such that $\gamma(G-X) = \gamma(G)$. This concept was introduced by V.R. Kulli *et al.* [7].

2. Interval Graph

Let I = {1,2,……...,n} be an interval family where each i in I is an interval on the real line and i = $[a_i, b_i]$ for i = 1, 2,…. n. Here a_i is called the left endpoint and b_i is called the right endpoint of i. Without loss of generality, we assume that all endpoints of the intervals in I are distinct numbers between 1 and 2n. Two intervals i and j are said to **intersect** each other if they have non-empty intersection. Two intervals are said to overlap if they have non-empty intersection and neither one of them contains the other.

Let G (V, E) be a graph. G is called an interval graph if there is a one-to-one correspondence between V and I such that two vertices of G are joined by an edge in E if and only if their corresponding intervals in I intersect.

Let G be the interval graph corresponding to the interval family I. Let nbd[i] be defined as the set of vertices adjacent to i including i. Let min(i) denote the smallest interval in nbd[i] and max (i) denote the largest interval in nbd[i]. Define Next (i) = j if and only if $b_i < a_j$ and there does not exist an interval k such that $b_i < a_k < a_j$. If there is no such j, we define Next (i) = null.

3. Algorithm : MDS - IG

Input : Interval family I = {1,2,… n}.
Output : Minimum dominating set of the interval graph G.

Step 1 : Let S = {max (1)}.
Step 2 : LI = The largest interval in S.
Step 3 : Compute Next (LI).
Step 4 : If Next (LI)=null then go to step.8
Step 5 : Find max(Next (LI)).
Step 6 : If max(Next (LI)) does not exist then max(Next (LI))=Next (LI).
Step 7 : S = S ∪ max(Next (LI))) go to step 2.
Step 8 : End.

4. Main Results

4.1 Co - Bondage Number

Theorem 1 : Let D = $\{x_1, x_2 \ldots\ldots x_m\}$ be such that $<N[x_1]-u_1>$,$<N[x_2]-u_2>$, ……......... $<N[x_m]>$ are cliques of size 3, where $u_1, u_2 \ldots\ldots u_{m-1}$ are the last vertices dominated by $x_1, x_2 \ldots\ldots x_{m-1}$ and also the first vertices dominated by $x_2, x_3 \ldots\ldots x_m$ respectively. Then

i) $b_c = 2$, if $\gamma = 2$.

ii) $b_c = 1$, if $\gamma > 2$.

Proof : Let $x_1, x_2, \ldots x_m$, $u_1, u_2, \ldots u_{m-1}$ satisfy the hypothesis of the theorem.

Case 1 : Suppose γ = 2. Let D = $\{x_1, x_2\}$ be a dominating set of G satisfying the hypothesis.

- *Dept. of Mathematics, SPWD & PG College, Tirupati .*
  E-mail: kalapatidhanalakshmi1@gmail.com
  Dept. of Applied Mathematics, SPMVV, Tirupati.
  E-mail: maherahul.55@gmail.com

ISSN 2229-5518

Consider $E_1 = \{(x_1, x_2 -1), (x_1, x_2)\}$. Now x_1 dominates all vertices of $N[x_1]$ and $N[x_2]$. Hence the dominating set in $G + E_1$ is $\{x_1\}$ only so that $\gamma(G+E_1)=1$.

We now prove that if we add a single edge e to G then $\gamma(G + e) < \gamma(G)$. Join the vertices (x_1, x_2-1). Let $e = (x_1, x_2-1)$. Then x_1 dominates u_1, $x_2 - 1$ in G + e. But x_1 does not dominate x_2, since x_2 is the last vertex in $N[x_2]$. Hence $D = \{x_1, x_2\}$ is also a dominating set in G + e so that $\gamma(G + e) < \gamma(G)$.

For all other possibilities of addition of a single edge to G, we can see that $\gamma(G + e) < \gamma(G)$.

Therefore $\gamma(G + E_1) < \gamma(G)$. Thus $b_e = 2$.

Case 2 : Suppose $\gamma > 2$.

Join the vertices x_1 and x_2-1. Let $e = (x_1, x_2-1)$. Consider the graph G + e. In this graph consider the vertex subset $S = \{x_1, u_2, u_3, ... u_{m-1}\}$. Since D is constructed by the algorithm, it is clear that u_1 is the last vertex dominated by x_1 and u_1 is the first vertex dominated by x_2. Since x_1, $x_2 - 1$ are joined by an edge, x_1 dominates x_2 -1 also. That is x_1 dominates all vertices that are dominated by x_2 except u_2.

Since u_2 is the last vertex dominated by x_2, it is clear that $u_2 \in < N[x_3] - u_3 >$. That is u_2 dominates all these vertices in this clique. Likewise u_i dominates all vertices in $< N[x_{i+1}] - u_{i+1} >$ where $i = 2, 3, \ldots$ m-2. It is obvious that u_{m-1} is the first vertex dominated by x_m and hence u_{m-1} dominates all vertices in $< N[x_m] >$. Thus the set S dominates all vertices in G + e. Hence $\gamma(G+e) \le |S| = m-1$.

But $\gamma(G) = m$, since D is the minimum dominating set constructed by the algorithm. Therefore $\gamma(G+e) \le |S| = m-1 < m = |D| = \gamma(G)$.

Thus $b_c(G) = 1$.

Theorem 2 : Let $D = \{x_1, x_2 \ldots x_m\}$ be such that $< N[x_1]-u_1 >, < N[x_2]-u_2 >, \ldots < N[x_m] >$ are cliques of size r, where $u_1, u_2, \ldots u_{m-1}$ are the last vertices dominated by $x_1, x_2, \ldots x_{m-1}$ and also the first vertices dominated by $x_2, x_3 \ldots x_m$ respectively. Then

i) $b_c = r - 1$ if $\gamma = 2$.

ii) $b_c = r - 2$ if $\gamma > 2$.

Proof : Suppose $\gamma = 2$. Let $D = \{x_1, x_2\}$ be a dominating set of G satisfying the hypothesis. For $x_1 \in D$, we draw additional edges between x_1 and all vertices in the consecutive clique $< N[x_2] - u_1 >$. As there are r vertices in any clique and we are joining x_1 to all vertices of $< N[x_2] -u_1 >$, there are r - 1 new edges added to the graph. Now the proof follows on similar lines to case 1 of Theorem 1 and hence $b_c = r-1$, if $\gamma = 2$.

Suppose $\gamma > 2$. For any $x_i \in D$, we draw additional edges between x_i and all vertices in the consecutive clique $< N[x_{i+1}]-u_i >$, except the vertex x_{i+1}. As there are r vertices in any clique and we are joining x_i to all vertices of $< N[x_{i+1}]-u_i >$, except x_{i+1} there are r - 2 new edges added to the graph. Now the proof follows on similar lines to case 2 of Theorem 1 and hence $b_c = r - 2$ if $\gamma > 2$.

Note : When we are adding additional edges in G by joining non – adjacent vertices, say u, v where u < v, then we are extending the left endpoint of v such that $a_v < b_u$, since right endpoint labelling of vertices was already done.

4.2 Efficient Bondage Number

Theorem 1 : Let $S = \{x_1, x_2 \ldots x_m\}$ be an efficient dominatingset of G such that $< N[x_1] >, < N[x_2] >, \ldots < N[x_m]>$ are cliques of size 3. Then $b_e = 2$.

Proof : Let $S = \{x_1, x_2 \ldots x_m\}$ be an efficient dominating set of G such that $< N[x_1] >, < N[x_2] >, \ldots < N[x_m] >$ are cliques of size 3. Let $u_1, \ldots u_m$ be the last vertices dominated by $x_1, x_2 \ldots x_m$ respectively.

Since $< N[x_i] >$'s are cliques of size 3, we observe that for $< N[x_1] >$, $x_1 = 1$ or 2 where $u_1 = 3$. For other cliques, obviously x_i's are middle vertices, since any vertex in $V \setminus S$ is dominated by exactly one vertex in S.

Let $f = (x_1-1, x_1)$. Consider the graph G - f. We now construct an efficient dominating set S_1 in G - f as follows. Since $N[x_1]$ is a clique, clearly u_1 dominates x_1-1, x_1. So we take u_1 into efficient dominating set of G - f. Now the vertex x_2 can not be included into efficient dominating set of G - f, since the first vertex dominated by x_2 is dominated by u_1. So we include x_2 - 1 into efficient dominating set of G - f. Now we remain the vertices $x_3, x_4 \ldots x_m$ as it is. So the efficient dominating set in G - f is $S_1 = \{u_1, x_2-1, x_3 \ldots x_m\}$.

Obviously S_1 is an efficient dominating set of G, since S is an efficient dominating set of G and the vertices in S_1 are all vertices of S, except x_2-1, u_1. But as for the above discussion, no vertex of G - f is adjacent to both x_2-1 and u_1. Therefore S_1 is an efficient dominating set of G - f.

Now the cardinality of this efficient dominating set in G - f is the same as S. Thus deletion of a single edge in G will not improve the cardinality of an efficient dominating set of G. Thus $b_e \neq 1$. Similar is the case if we deal with any other single edge in respective cliques.

Let $E_1 = \{(x_1-1, x_1),(x_1, u_1)\}$. Consider G - E_1 . Since $N[x_1]$ is a clique of size 3, clearly x_1 becomes isolated and thus included into efficient dominating set of G - E_1 as $S_1 = \{x_1, u_1, x_2-1, x_3 \ldots x_m\}$.

ISSN 2229-5518

Since S is minimum and the insertion of u_1, x_2-1 into S_1 is essential because these vertices dominate the vertices that precede x_3 and x_1 is isolated, it follows that S_1 is minimum in G - E_1.

Hence $\left|S_1\right| = m - 1 + 2 = m + 1$.

Therefore

$$\gamma_e\ (G - E_1) = \left|S_1\right| = m + 1 > m = \gamma_e\ (G)\ .$$

Thus $b_e = 2$.

Theorem 2 : Let S = {x_1, x_2. ………. x_m} be an efficient dominatingset of G such that $< N[x_1] >, < N[x_2] >$……. $< N[x_m] >$ are cliques of size 4 or 5. Then $b_e = 3$.

Proof : Let S = {x_1, x_2……………….. x_m} be an efficient dominating set of G such that $< N[x_1] >$,……….……. $< N[x_m] >$ are cliques of size 4. Let u_1, u_2 ………… u_m be the last vertices dominated by x_1, x_2 ……… x_m respectively.

Since $< N[x_i] >$'s are cliques of size 4, we observe that for $< N[x_1] >$, x_1 = 1 or 2 or 3 where u_1 = 4. Let x_1 = 2. By Theorem 1, we have $b_e \neq 1$. Let E_1 = {(x_1-1, x_1), (x_1+1, u_1)}. Consider G - E_1. We now construct an efficient dominating set S_1 in G - E_1 as follows. We give a selection of vertices in the first two cliques and we remain the vertices in other cliques as it is. Clearly x_1+1 dominates x_1-1, x_1. So x_1+1 is included into efficient dominating set of G - E_1. Now we can not take u_1 into efficient dominating set of G - E_1, since x_1 -1 is adjacent to both u_1 and x_1+1. So we include u_1+1 into efficient dominating set of G - E_1, where u_1 is dominated by u_1+1.Here we note that $x_2 \neq u_1+1$. Otherwise u_1 is dominated by both u_1+1 and x_1, a contradiction to the fact that S is an efficient dominating set of G.

Let S_1 = {x_1 +1, u_1+1, x_3 ……………x_m}. Clearly S_1 is an efficient dominating set of G and the cardinality of this efficient dominating set in G - E_1 is the same as that of S. Since we have not disturbed the structure of cliques except the first one, and a single vertex only required to dominate the vertices in $< N [x_1] >$ in G - E_1, it follows that there will not be any efficient dominating set of G - E_1 with lower cardinality than S_1. Thus deletion of two edges in G also does not increase the cardinality of an efficient dominating set of G - E_1. Thus $b_e \neq 2$.

For other choices of x_1 in the first clique and also choice of any two edges in other cliques, which include the choice of the edge (u_i, u_i+1) does not increase the cardinality of efficient dominating set of G - E_1.

Let E_2 = {(x_1-1, x_1), (x_1, x_1+1), (x_1+1, u_1)}. Consider G - E_2. We now construct an efficient dominating set S_2 in G - E_2 as follows. We can not take both the vertices x_1-1, x_1 into S_2, because they dominate u_1. So choose either x_1-1 or x_1 into S_2. Let us choose $x_1 \in S_2$. Now x_1 dominates u_1. In order that x_1-1, x_1+1 are dominated, we choose x_1+1 into S_2. Now we remain the vertices x_2, x_3 ……..… x_m as it is. So S_2 = {x_1, x_1+1, x_2, x_3…. x_m}.

Clearly S_2 is an efficient dominating set of G - E_2. Further by the selection of the vertices into S_2, it is clear that there will not be any efficient dominating set of G - E_2 with lower cardinality than that of S_2.

Therefore
$$\gamma_e(G - E_2) = |S_2| = m + 1 > m = |S| = \gamma_e\ (G).$$

Hence b_e = 3. Now consider cliques of size 5. As in the previous discussion, we can see that deletion of any 2 edges from G, will not increase the cardinality of efficient dominating set in the resultant graph. Hence consider E_1 = {(x_1, u_1), (x_1, x_1+1), (x_1-2, x_1-1)} where x_1= 3, u_1 = 5.

Consider the induced subgraph $<N[x_1]>$ in G - E_1. In this graph, no single vertex can dominate the other vertices. So we select 2 vertices, say x_1-1, x_1 into efficient dominating set of G - E_1.

Let S_1 = {x_1-1, x_1, x_2, x_3 ……. x_m}. Clearly this is an efficient dominating set of G - E_1 and we can easily see that there is no efficient dominating set of lower cardinality than S_1 in G - E_1.

Hence
$$\gamma_e(G - E_1) = \left|S_1\right| = m + 1 > m = |S| = \gamma_e\ (G).$$

$\therefore b_e\ (G) = 3.$

We now generalise the above result as follows.

Theorem 3 : Let S = {x_1, x_2 ……..…….x_m} be an efficient dominating set of G such that $< N[x_1] >$, ………… $< N[x_m] >$ are cliques of size r, r > 5. Then b_e = r - 2.

Proof : Proof follows on similar lines to that of Theorem 2. Here in any clique N[x_1], we delete r - 2 edges.

Theorem 4 : Let S = { u,v} be a minimum efficient dominating set of G. Then b_e = 1.

Proof : Let S = {u, v} be an efficient dominating set of G. Since S is an efficient dominating set of G and we have right endpoint labelling, it is clear that the vertices preceeding to u will be dominated by u and suceeding vertices of u are dominated by v.

Consider the edge f = (u-1, u) and the graph G - f. In this graph u-1 is not dominated by u. Obviously u-1 is not dominated by v also. Then we have the following cases.

Case 1 : There may be a vertex y < u - 1 such that y dominates u - 1. Suppose there is no other vertex t < u - 1 such that t also dominates u - 1. Consider S_1 = {y, u, v}. Clearly S_1 is an efficient dominating set in G - f and hence b_e = 1.

ISSN 2229-5518

Sub case 1 : Suppose there is a vertex $t < u - 1$ such that t also dominates u-1. Again $S = \{y, u, v\}$ is an efficient dominating set of G - f. Hence $b_e = 1$.

Sub case 2 : Suppose there is no vertex $y < u-1$ such that y dominates u -1. Then u-1 is an isolated vertex in G - f. Hence $S_1 = \{u - 1, u, v\}$ is a minimum efficient dominating set in G - f and thus $b_e = 1$.

We have deleted an edge that is adjacent with u and proved that $b_e = 1$. Similar is the case if we replace u by v or if we argue with both u and v.

4.3. Non - Bondage Number

Theorem 1 : If G is an interval graph then $b_n(G) = q - p + \gamma(G)$, where p, q are the number of vertices and edges in G.

Proof : Let D be the minimum dominating set of G constructed by the Algorithm. Let v be any vertex of G. Consider an edge that is adjacent with v and a vertex of D. Likewise consider the set E_1 of all edges that are incident with vertices of G and vertices of D. Then there are $p - \gamma$ such edges.

Consider $E - E_1$. Then $\left|E - E_1\right|$ $= q - (p - \gamma) = q - p + \gamma$. Let $X = E - E_1$. Consider the graph $G - X = G - E + E_1$. This graph contains only the edges that are in E_1. That is the set of edges which are incident with the vertices of D. Therefore the dominating set of this graph is nothing but the set D.

Hence $\gamma(G - X) = \gamma(G)$.

Thus $b_n(G) = \left|X\right| = \left|E - E_1\right| = q - p + \gamma$. The proof of the following corollary is immediate.

Corollary

Let G be an interval graph on n vertices such that G is a path. Then $b_n(G) = \gamma - 1$.

5. Illustrations

Cobondage Number

Theorem 1

Case 2 :

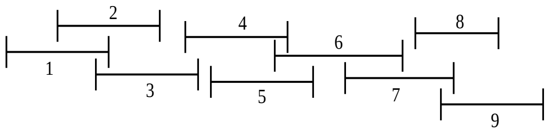

Fig. 1 : Interval Family I

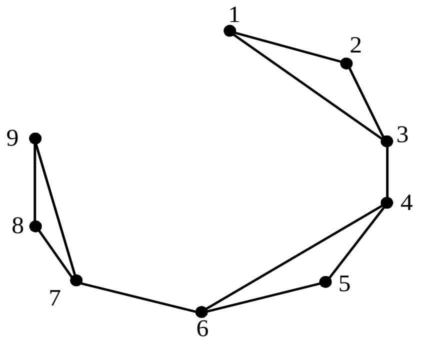

Fig. 2: Interval Graph G

The dominating set according to the algorithm is $D = \{3, 6, 9\}$.

Here $x_1 = 3, x_2 = 6, x_3 = 9, u_1 = 4, u_2 = 7$

Let $e = (x_1, x_2 - 1) = (3,5)$

Consider the graph G + e

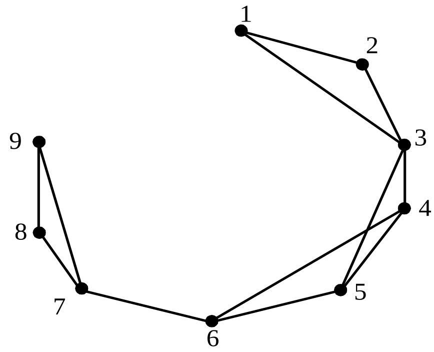

Fig .3 : G +e

The dominating set in G + e is $D_1 = \{3,7\}$

Clearly $\gamma(G + e) = 2 < \gamma(G)$

Thus $b_c = 1$.

Efficient bondage number

Theorem 4

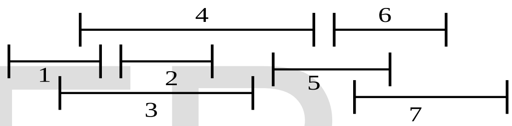

Fig. 1 : Interval Family I

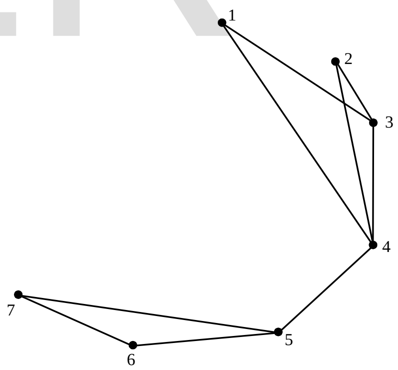

Fig. 2 : Interval Graph G

The efficient dominating set in G is $S = \{4, 5\}$

Here $u = 4, v = 5, y = 2, u-1 = 3, t = 1$

Consider the edge, $f = (u - 1, u) = (3,4)$

Consider G - f

ISSN 2229-5518

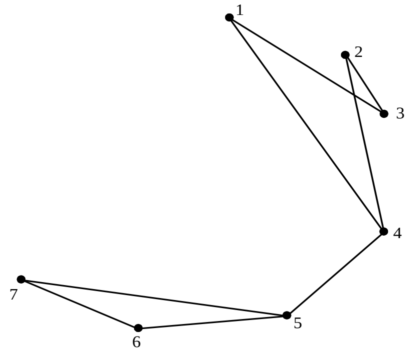

Fig. 3 : G - f

The efficient dominating set in G - f is {2, 4, 5}
Thus $b_e = 1$.

Nonbondage Number
Theorem 1

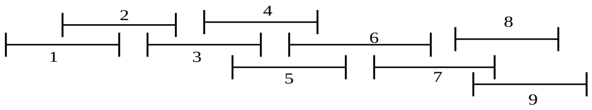

Fig. 1 : Interval Family I

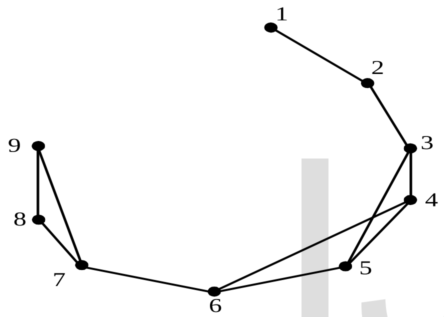

Fig. 2 : Interval Graph G

ISSN 2229-5518

The minimum dominating set according to the algorithm is D = {2, 6, 9}

Here $V \setminus D$ = {1, 3, 4, 5, 7, 8}. Here $\gamma = 3$, $p = 9$, $q = 11$

Let E_1 = {(1, 2), (2, 3), (4, 6), (5, 6), (7, 9), (8, 9)}

$E \setminus E_1$ = {(3, 4), (3, 5), (4, 5), (6, 7), (7, 8)}

Clearly $X = \left| E - E_1 \right| = q - (p - \gamma) = 11 - (9 - 3)$

$= 5$

Consider $G - X = G - E + E_1$

= {(1, 2), (2, 3), (4, 6), (5, 6), (7, 9), (8, 9)}

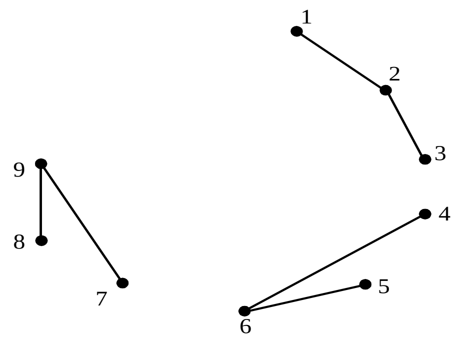

Fig. 3 : Interval Graph $G - E + E_1$

The dominating set in this graph is {2, 6, 9} = D only.

Hence $b_n(G) = \left| X \right| = 5$.

6. REFERENCES

[1] O. Ore "Theory of Graphs", Amer. Math. Soc. Colloq. Publ. 38, Providence, 1962.

[2] C.Berge "Theory of Graphs" and its Applications, Methuen, London, 1962.

[3] E.J.Cockayne S.T.Hedetniemi-Towards a theory of domination in graphs, Networks 7 , 1977, 247-261.

[4] V.R.Kulli B.Janakiram- The cobondage number of a graph, Discussions Mathem-aticae, Graph Theory 16, 1996, 111-117.

[5] E.J.Cockayne B.L.Hartnell S.T.Hedet-niemi R.Laskar- Efficient domination in graphs, 1988, Clemson Univ., Dept. of Mathematical Sciences, Tech. Report, 558.

[6] V.R.Kulli N.D.Soner -Efficient bondage number of a graph. Nat. Acad.Sci. Letters, Vol.19, No.9 and 10, 1996.

[7] V.R.Kulli B.Janakiram -The nonbondage number of a graph, Graph Theory Notes of New York, XXX:4 New York Academy of Sciences, 1996, 14-16.